Def: An (infinite) sequence is an ordered list of numbers of the form

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

Ex 1

$$0, 2, 4, 6, 8, 10, \ldots$$

a_0 a_1 a_2 a_3 a_4 a_5

Notation: We can denote a sequence in the following ways:

(a) $\{a_n\}_{n=n_0}^{\infty}$ → fixed, indicates first index

(b) $(a_n)_{n=n_0}^{\infty}$ →

(c) "$a_n = f(n)$ for $n \geq n_0$" where f is some function

explicit form

(d)
- Specify a_{n_0}
- define a_{n+1} in terms of $a_{n_0}, \ldots, a_n$

recursive form

Ex 2 Write the sequence from example 1 in explicit and recursive form.

Explicit form: $a_n = 2n$ for $n \geq 0$

Recursive form:

$$a_0 = 0$$
$$a_{n+1} = a_n + 2$$

Def: For the sequence, a_n, the variable n is called the index variable

Note: We don't always use n for the index variable. Sometimes we use m, k, i, j instead.

Ex 3 Find an explicit formula for the sequence $\{7, 23, 39, 55, 71, 87, \dots\}$.

a_0 a_1 a_2

16 16 16 16 16

a_1 a_2 a_3

7, 16, 23, ...

① Pattern

② $a_n = 7 + 16n$ for $n \geq 0$

$a_n = 7 + 16(n-1)$ for $n \geq 1$

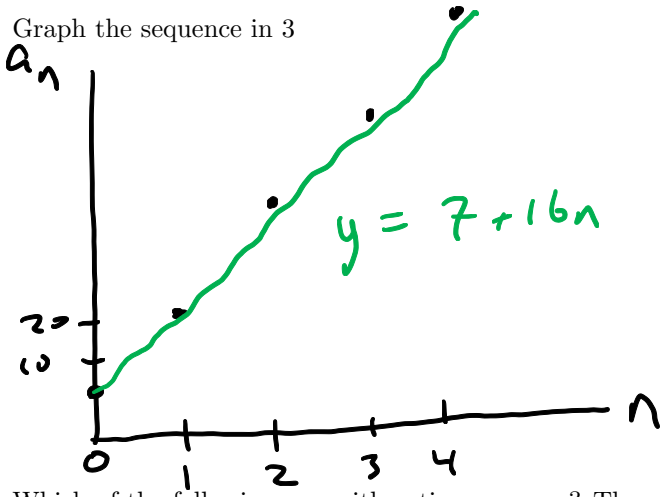
AIR-ith-MET-ic

Def: An arithmetic sequence is a sequence where the difference between every pair of consecutive terms is constant.

Thm: Every arithmetic sequence has the form

$$a_n = b + mn \quad \text{for } n \geq n_0$$

Ex 4 Graph the sequence in 3



PQ. Which of the following are arithmetic sequences? There may be more than one correct answer.

(a) $a_n = 2n + 3$ for all $n \geq 0$ (m, b)

~~(b)~~ $b_i = 2^i - 1$ for all $i \geq 0$

~~(c)~~ $c_j = \ln(3j + 5)$ for all $j \geq 0$

(d) $d_k = -k$ for all $k \geq 0$ ($-1 = m$, $0 = b$)

$b_0 = -1$ $b_1 = 1$ $b_2 = 3$ $b_3 = 7$ (4)

Ex 5 Find an explicit formula for the sequence defined by the recurrence relation $f_0 = 1$ and $f_{n+1} = \dfrac{f_n}{n+1}$ for all $n \geqslant 1$.

$f_0 = 1$

$f_1 = \dfrac{f_0}{1} = 1$ ($n+1 = 1$, $n = 0$)

$f_2 = \dfrac{f_1}{2} = \dfrac{1}{2}$ ($n+1 = 2$, $n = 1$)

$f_3 = \dfrac{f_2}{3} = \dfrac{1}{2 \cdot 3}$ ($n+1 = 3$, $n = 2$)

$f_4 = \dfrac{f_3}{4} = \dfrac{1}{2 \cdot 3 \cdot 4}$ ..... $f_n = \dfrac{1}{1 \cdot 2 \cdots n} = \dfrac{1}{n!}$

Ex 6 Find an explicit formula for the sequence defined by $x_0 = 3$ and $x_{n+1} = -5x_n$ for all $n \geqslant 1$.

$\dfrac{x_{n+1}}{x_n} = -5$

the next term is -5 times the previous term

$x_0 = 3 = 3$

$x_1 = (-5) \cdot 3 = -15$

$x_2 = (-5)^2 \cdot 3 = 75$

$x_3 = (-5)^3 \cdot 3 = -375$

$\vdots$

$x_n = (-5)^n \cdot 3$

Def: A geometric sequence is a sequence where the ratio between every pair of consecutive terms is constant.

Thm: Every geometric sequence has the form

$a_n = b \cdot r^n$ for constants b, r

PQ. Which of the following defines a geometric sequence? There may be more than one correct answer.

(a) $a_n = 2 \cdot 3^n$ for all $n \geqslant 0$

(b) ~~$b_n = 2 + 3n$ for $n \geqslant 0$~~

(c) $c_n = 3\cos(\pi n)$ for all $n \geqslant 0$

(d) ~~$d_n = \frac{1}{2n+7}$ for all $n \geqslant 0$~~

(e) ~~$e_n = 4 \cdot n^3$ for all $n \geqslant 0$~~

(f) $f_n = -2 \cdot 4^{n+1}$ for all $n \geqslant 0$

$= 3 \cdot (-1)^n$

$c_0 = 3$) $\cdot(-1)$
$c_1 = -3$) $\cdot(-1)$
$c_2 = 3$) $\cdot(-1)$
$c_3 = -3$) $\cdot(-1)$
$c_4 = 3$
$\vdots$

$(-1)^{\text{even \#}} = 1$

$(-1)^{\text{odd \#}} = -1$

$d_0 = 1/2 \quad d_1 = 1/4 \quad d_2 = 1/11$

$7/4 \qquad 9/11$

$e_0 = 0, \; e_1 = 4, \; e_2 = 32, \; e_3 = 108$

$-$ $\qquad$ 8 $\qquad$ $\neq 8$

Def: The Fibonacci sequence $(F_n)_{n=0}^{\infty}$ is defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}$$

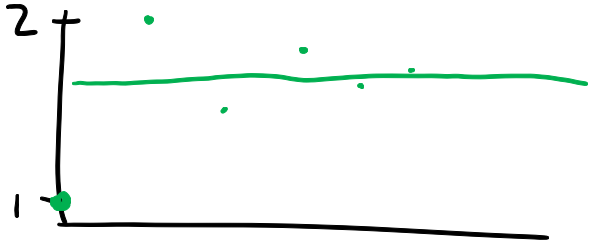
Ex 7 Write out the first eight terms of the Fibonacci sequence and decide if $(F_n)_{n=0}^{\infty}$ is arithmetic, geometric, or neither.

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13

Diff: 1 0 1 1 2 3 5

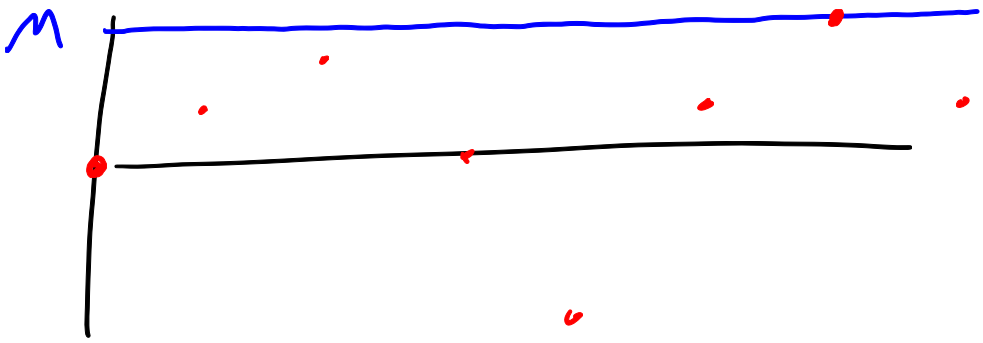
Ratio: – 1 2 1.5 1.67 1.6 1.625 → Golden Ratio φ $\frac{1+\sqrt{5}}{2}$

Neither

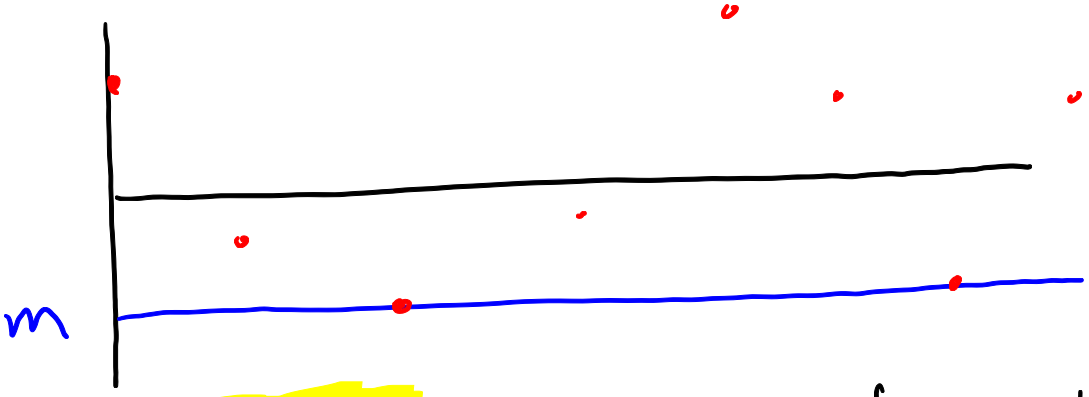


Def: Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence.

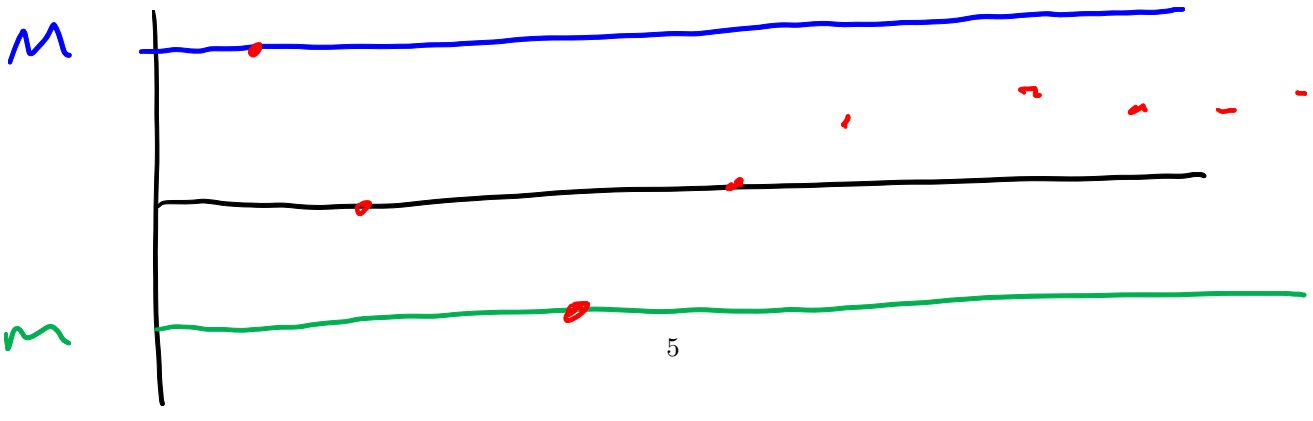
(a) $\{a_n\}_{n=0}^{\infty}$ is bounded from above if there exists a number M such that $a_n \leq M$ for all n.



(b) $\{a_n\}_{n=0}^{\infty}$ is bounded from below if there exists a number m such that $a_n \geq m$ for all n.

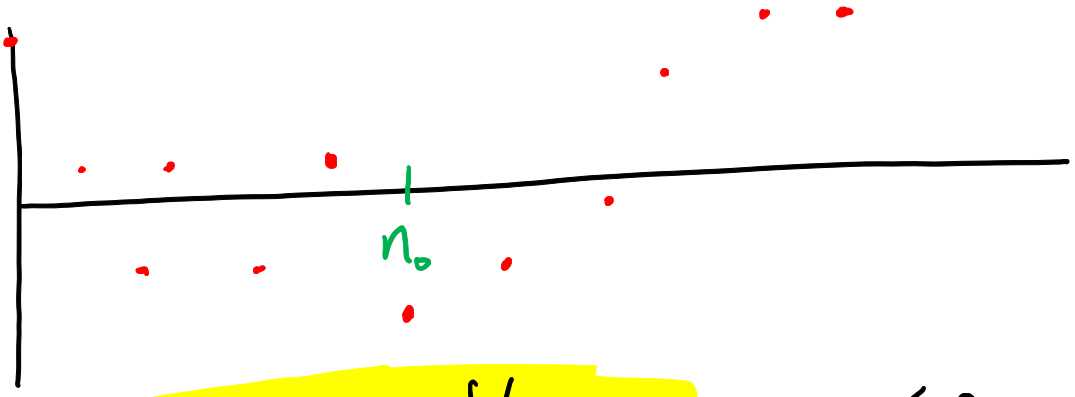


(c) $\{a_n\}_{n=0}^{\infty}$ is bounded if it is bounded from above and below



n zero OR n naught

(d) $\{a_n\}_{n=0}^{\infty}$ is increasing after n_0 if $a_n \geq a_{n-1}$ for all $n \geqslant n_0$. We might also say that $\{a_n\}_{n=0}^{\infty}$ is eventually increasing



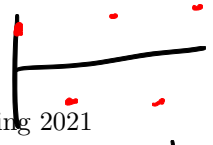
(e) $\{a_n\}_{n=0}^{\infty}$ is decreasing after n_0 if $a_n \leq a_{n-1}$ for all $n \geqslant n_0$. We might also say that $\{a_n\}_{n=0}^{\infty}$ is eventually decreasing



(f) $\{a_n\}_{n=0}^{\infty}$ is monotone after n_0 (or eventually monotone) if it is either increasing after n_0 or decreasing after n_0.

- either of the above pictures

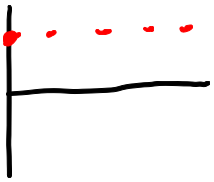
PQ: Can a sequence be both inc. and dec? A. Yes B. No



neither

$a_n = 2$

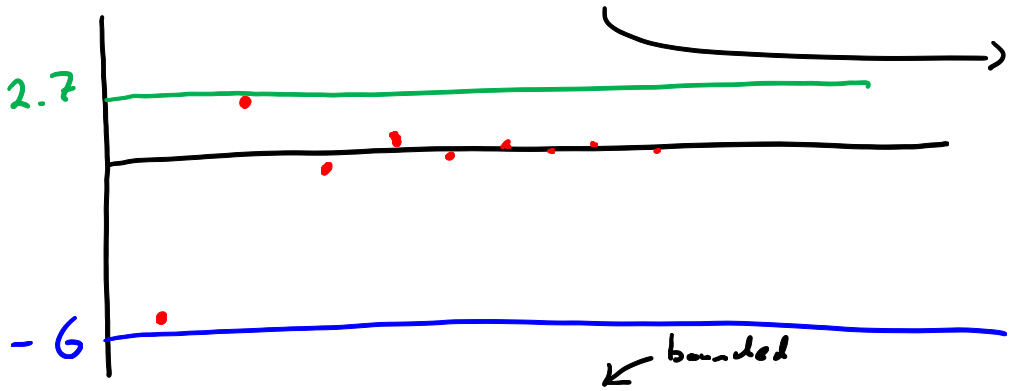
Ex 8 Recall the Fibonacci Sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \geqslant 1$. Is $\{F_n\}_{n=0}^{\infty}$ decreasing for all $n \geqslant 0$ or increasing for all $n \geqslant 0$ (or neither)? Is $\{F_n\}_{n=0}^{\infty}$ bounded below, above, or neither?



- bounded below?
  - ↳ yes: bounded below by 0 because $F_n \geq 0$ for all n
- increasing after 0?
  - ↳ yes: $F_{n+1} = F_n + \boxed{F_{n-1}} \geq F_n$ for all $n \geq 0$ ($F_{n-1} \geq 0$)
- decreasing after 0?
  - ↳ $F_{n+1} = F_n + \boxed{F_{n-1}} > F_n$ for $n > 1$ ($F_{n-1} > 0$ for $n > 1$), so not decreasing
- bounded above?
  - ↳ no; $F_{n+1} \geq F_n + 1$

bounded above and below?

Ex 9 Consider the geometric sequence $\{y_n\}$ defined by $y_n = 10 \cdot \left(-\frac{1}{2}\right)^n$ for all $n \geqslant 1$. Is $\{y_n\}$ bounded? Is there an index n_0 such that $\{y_n\}$ is monotone for all $n \geqslant n_0$?



The sequence is bded. above by 2.7 and below by -6, so it is bded.

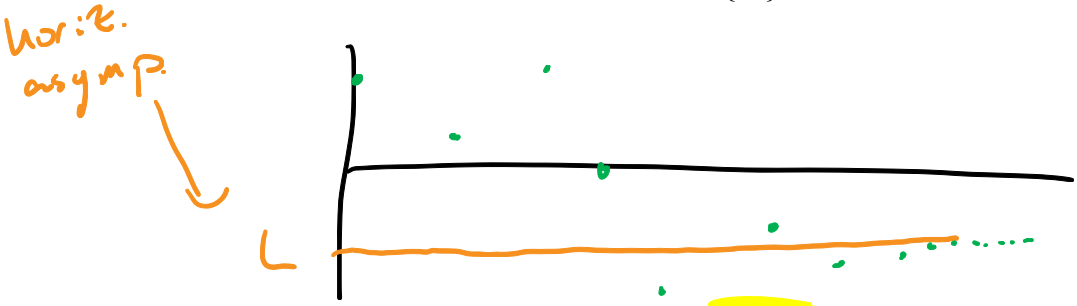
y_n is not monotone/monotonic

don't use on exams

Def: (Informal) We say the sequence $\{a_n\}$ converges to a number L if "a_n gets arbitrarily close to L" for "sufficiently large" values of n. In this case, we call $\{a_n\}$ a convergent sequence and we write

$$\lim_{n\to\infty} a_n = L$$

and we call L the limit of $\{a_n\}$.



Def: (Formal) We say the sequence $\{a_n\}$ converges to the limit L if

epsilon ϵ (small)

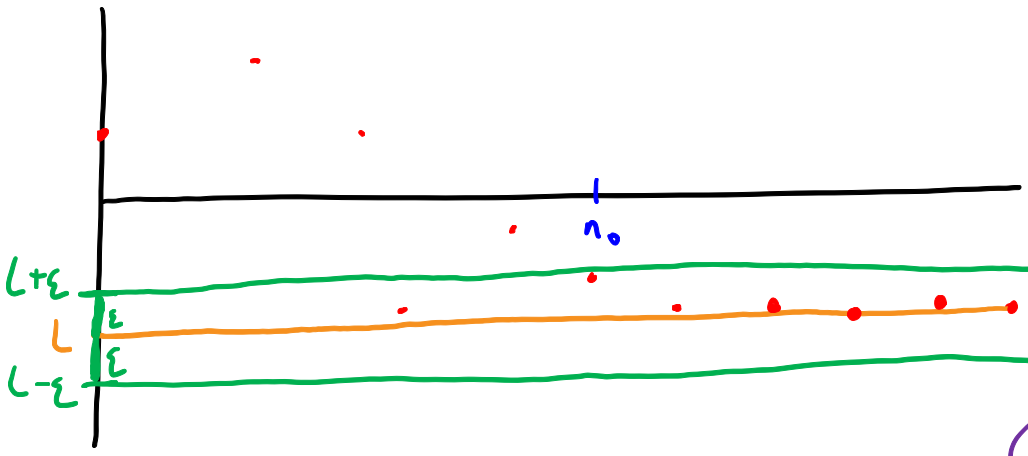
for every $\varepsilon > 0$, there exists n_0 so that for all $n \geq n_0$, $|a_n - L| < \varepsilon$.

for every positive distance

eventually

a_n is within ε of L

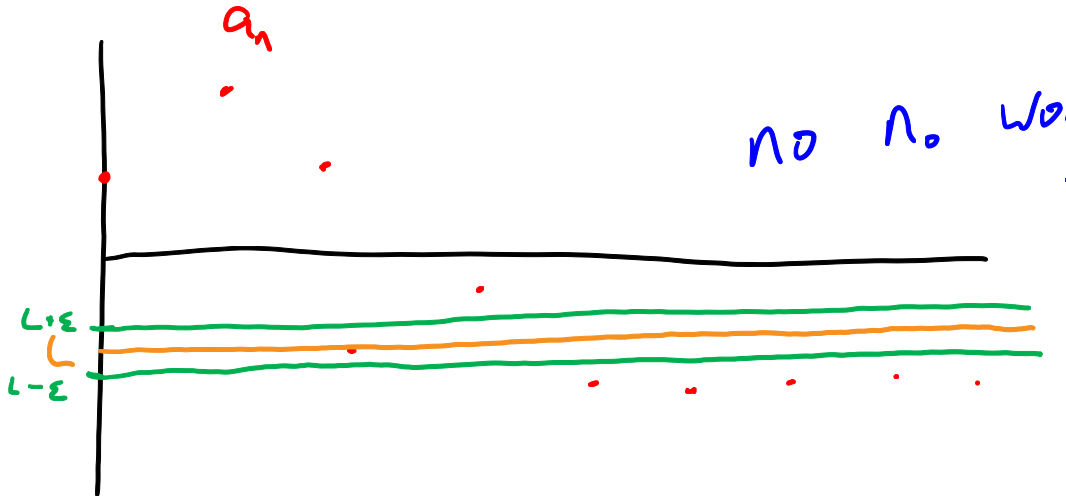
a_n is close to L



1. Graph a_n
2. Pick L
3. Pick ε
4. Find n_0 s.t. (so that) for every $n \geq n_0$, a_n is inside green lines
5. Go to (3) with smaller ε

Q. What does it look like for a sequence a_n to converge to L?

Q. What does it look like for a sequence a_n to converge to something other than L?



no n_0 works for this ε

$\lim_{n\to\infty} a_n \neq L$

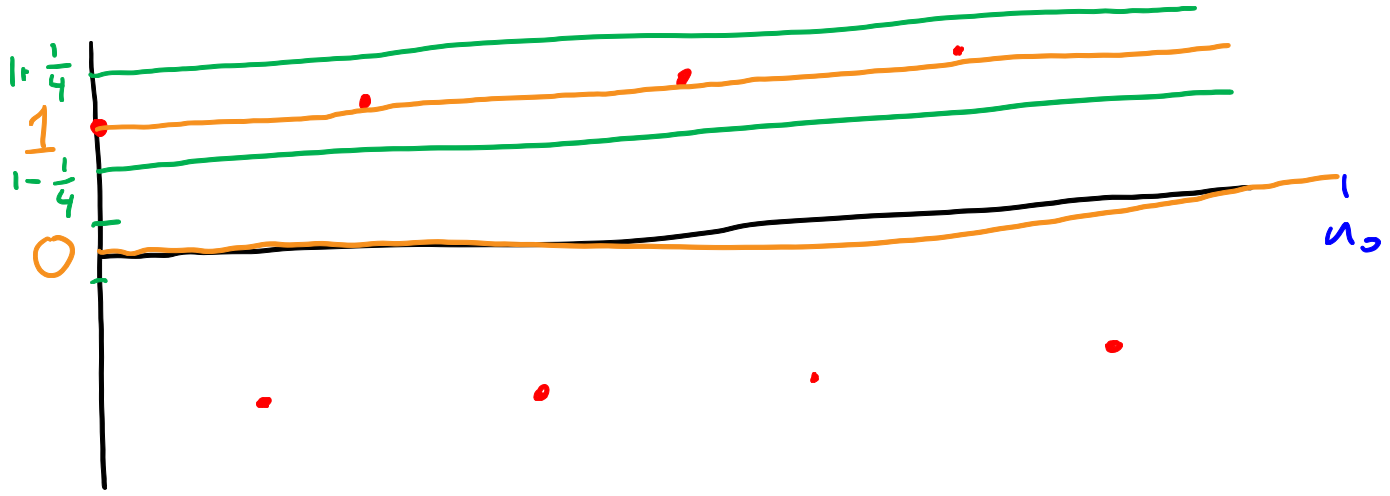
Pick a new L

Here's a second way of thinking about convergence. $\lim_{n\to\infty} a_n = L$ if we can have the following conversation:

- You: when does a_n come within ε (1) of L? (infinite)
- Me: For all $n \geq 10$, (n_0) a_n is within 1 of L.
- You: When does a_n come within 0.1 of L?
- Me: For all $n \geq 100$, a_n with 0.1 of L.
- You: When does a_n come within 0.01 of L?
- Me: For all $n \geq 1000$, a_n is within 0.01 of L.
- ⋮

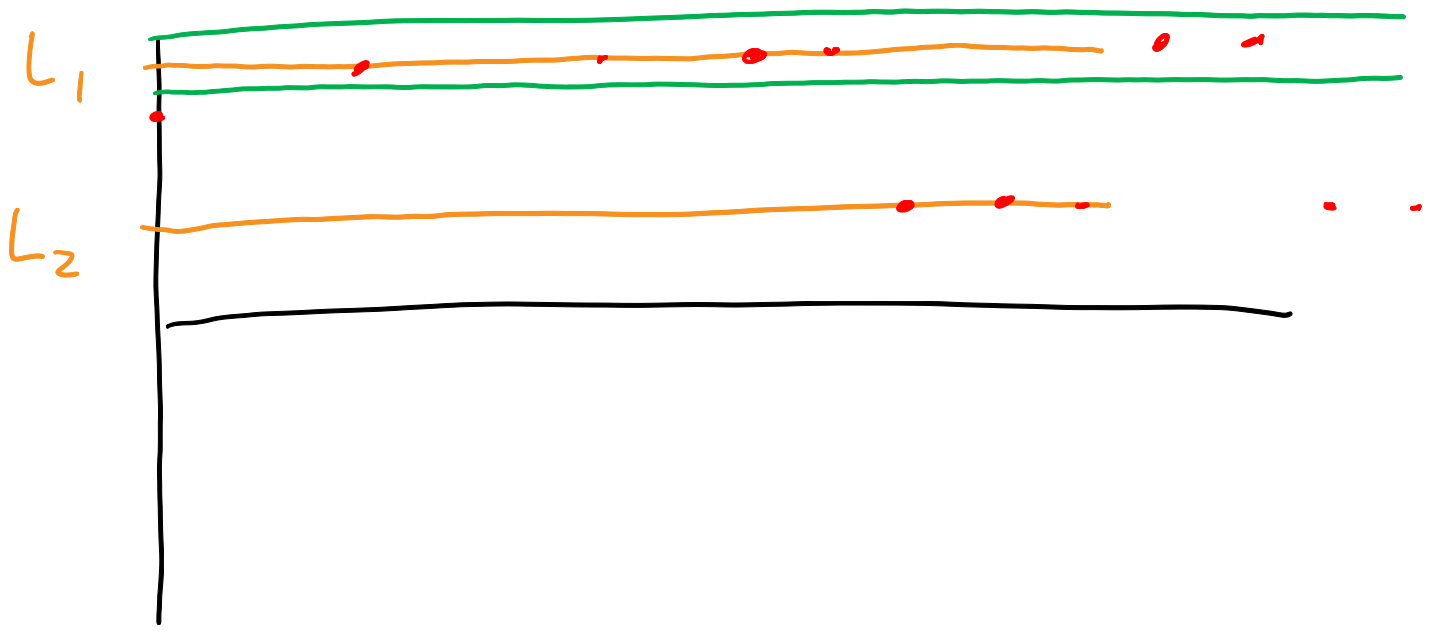
Q: What does a_n not having a limit look like?

$$a_n = (-1)^n$$

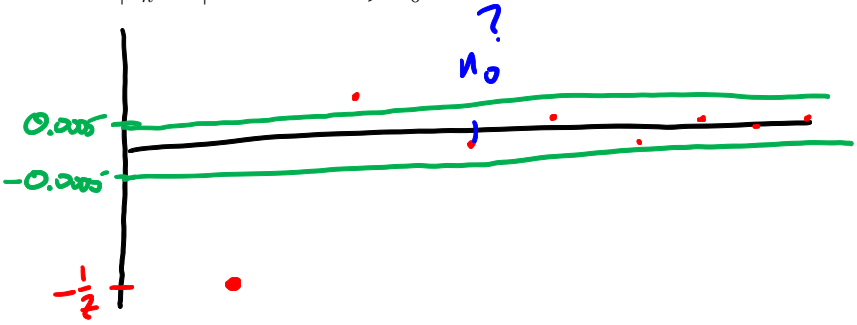


Q: Can a sequence have more than one limit?

A: N_0



Ex 10 Suppose $x_n = \left(-\frac{1}{7}\right)^n$ for all $n \geqslant 1$. Let $\varepsilon = 0.0005$. Find an explicit integer n_0 with the property that $|x_n - 0| < \varepsilon$ for all $n \geqslant n_0$.



Want: $|x_n| < 0.0005$ for all $n \geq n_0$

LHS: $|x_n| = \left|\left(-\frac{1}{7}\right)^n\right| = \left(\frac{1}{7}\right)^n = \frac{1^n}{7^n} = \frac{1}{7^n}$

"solve for n" → Want: $\frac{1}{7^n} < 0.0005 \leftrightarrow \frac{1}{0.0005} < 7^n \overset{*}{\leftrightarrow} \log_7\left(\frac{1}{0.0005}\right) < n$

$n \geq 3.9$

If $n \geq 4$, then $|x_n| < 0.0005$

n_0

* $x < y \leftrightarrow \log_b(x) < \log_b(y)$ if and only if $b > 1$

Ex 11 How do you convince someone $\lim_{n\to\infty} \frac{2n^2}{1+n^2} = 2$?

Find n_0 for $\varepsilon = 10^{-30}$

Want: $\left|\frac{2n^2}{1+n^2} - 2\right| < 10^{-30}$

Note: $\frac{2n^2}{1+n^2} = 2 \cdot \frac{n^2}{1+n^2} < 2$ ($\frac{n^2}{1+n^2} < 1$)

LHS: $\left|\frac{2n^2}{1+n^2} - 2\right| = 2 - \frac{2n^2}{1+n^2} = \frac{2(1+n^2)}{1+n^2} - \frac{2n^2}{1+n^2}$

$= \frac{2}{1+n^2}$

Want: $\frac{2}{1+n^2} < 10^{-30} \leftrightarrow 2 < (1+n^2)10^{-30}$

$\leftrightarrow 2 \cdot 10^{30} < 1+n^2$

$\leftrightarrow 2 \cdot 10^{30} - 1 < n^2$

$\leftrightarrow \sqrt{2 \cdot 10^{30} - 1} < n$

10

If $n > 10^{18}$, then $\left|\frac{2n^2}{1+n^2} - 2\right| < 10^{-30}$

Ex 12 Compute $\lim_{n\to\infty} r^n$ for any (constant) r.

$r = 1$: $\lim_{n\to\infty} 1^n = 1$

$r = 2$: $\lim_{n\to\infty} 2^n = \infty$

$2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^3 = 8, \ldots$

$r = -3$: $\lim_{n\to\infty} (-3)^n$ DNE

$(-3)^0 = 1,\ (-3)^1 = -3,\ (-3)^2 = 9,\ (-3)^3 = -27, \ldots$



see next page

$r = \frac{-1}{5}$: $\lim_{n\to\infty} \left(\frac{-1}{5}\right)^n = 0$



$\left(\frac{-1}{5}\right)^0 = 1,\ \left(\frac{-1}{5}\right)^1 = \frac{-1}{5},\ \left(\frac{-1}{5}\right)^2 = \frac{1}{25}, \ldots$

Def: If a sequence $\{a_n\}$ does not converge to any limit, we say $\{a_n\}$ diverges or is divergent

Note A sequence can diverge in the following ways:

(a) $\lim_{n\to\infty} a_n = \infty$

(b) $\lim_{n\to\infty} a_n = -\infty$

(c) $\lim_{n\to\infty} a_n$ DNE

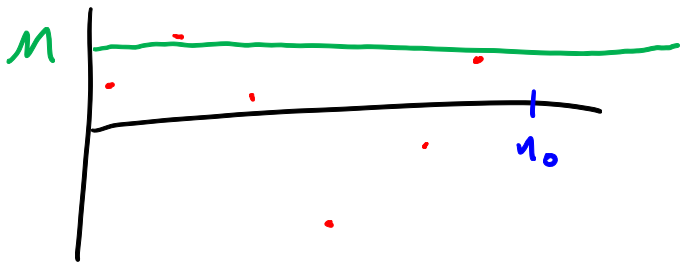
Ex 12)

$$\lim_{n\to\infty} r^n \begin{cases} \text{DNE} & r \le -1 \\ = 0 & -1 < r < 1 \leftrightarrow |r|<1 \\ = 1 & r = 1 \\ = \infty & r > 1 \end{cases}$$

Def: A sequence $\{a_n\}$ diverges to ∞ if

for every $M > 0$, there exists n_0 s.t. for all $n \geq n_0$, $a_n > M$

M large

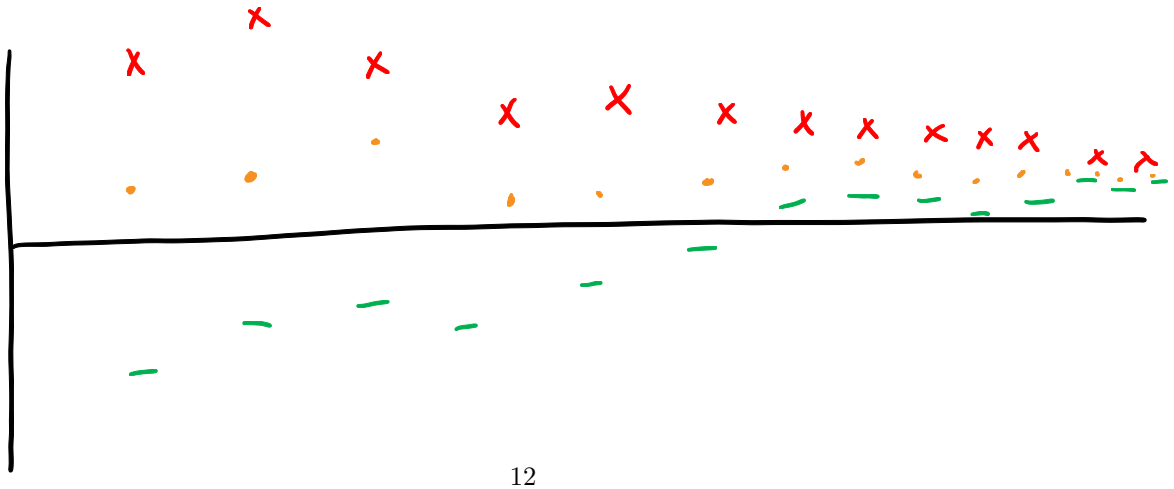


Def: A sequence $\{a_n\}$ diverges to $-\infty$ if

$\forall M < 0 \quad \exists n_0 \quad$ s.t. $\forall n \geq n_0, \; a_n < M$

for every — there exists — so that / such that

Thm: (Squeeze Theorem for Sequences) Suppose $a_n \leq b_n \leq c_n$ for all $n \geqslant n_0$ and suppose that $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$. Then $\lim_{n\to\infty} b_n = L$



Ex 13 Let $b_n = \dfrac{\sin(2^n)}{2^n}$. Compute $\lim_{n\to\infty} b_n$.

$$-1 \le \sin(2^n) \le 1$$

$$-\left(\frac{1}{2}\right)^n = \frac{-1}{2^n} \le \frac{\sin(2^n)}{2^n} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

$$0 = \lim_{n\to\infty} -\left(\frac{1}{2}\right)^n \le \lim_{n\to\infty} \frac{\sin(2^n)}{2^n} \le \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

$$\lim_{n\to\infty} \frac{\sin(2^n)}{2^n} = 0$$

Ex 14 Compute $\lim_{n\to\infty} \dfrac{n!}{n^n}$. But guess first!

PQ.
(a) 0
(b) 1
(c) ∞
(d) $-\infty$
(e) e
(f) π
(g) The limit does not exist

$$n! = n(n-1)(n-2)\cdots 1$$

$$n^n = n \cdot n \cdot n \cdots n$$

$$0 \le \frac{n!}{n^n} = \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n}}_{\le 1} \frac{1}{n} \le \frac{1}{n}$$

$$\frac{n}{n} = 1 \qquad \frac{n-1}{n} \le 1 \qquad \frac{n-2}{n} \le 1 \qquad \frac{2}{n} \le 1$$

$0 \le \frac{n!}{n^n} \le \frac{1}{n}$, with $0 \to 0$ and $\frac{1}{n} \to 0$.

Thm: (Algebraic Limit Laws) If $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, and c is a constant, then:

(a) $\lim_{n\to\infty} c = c$

(b) $\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n = cA$

(c) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$

(d) $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$

(e) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $b_n \neq 0$ for all n and $B \neq 0$.

Ex 15 Determine whether the sequence $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$ converges. If it converges, then compute the limit.

num $\to \infty$

denom $\to -\infty$

$\to$ (c)

$$\lim_{n\to\infty} \frac{3n^4 - 7n^2 + 5}{6 - 4n^4} = \lim_{n\to\infty} \frac{n^4\left(3 - \frac{7}{n^2} + \frac{5}{n^4}\right)}{n^4\left(\frac{6}{n^4} - 4\right)}$$

$$= \frac{\lim_{n\to\infty} 3 - \frac{7}{n^2} + \frac{5}{n^4}}{\lim_{n\to\infty} \frac{6}{n^4} - 4}$$

$$= \frac{3}{-4} \leftarrow \text{finite \#}$$

so the sequence converges

$a^2 - b^2 = (a+b)(a-b)$

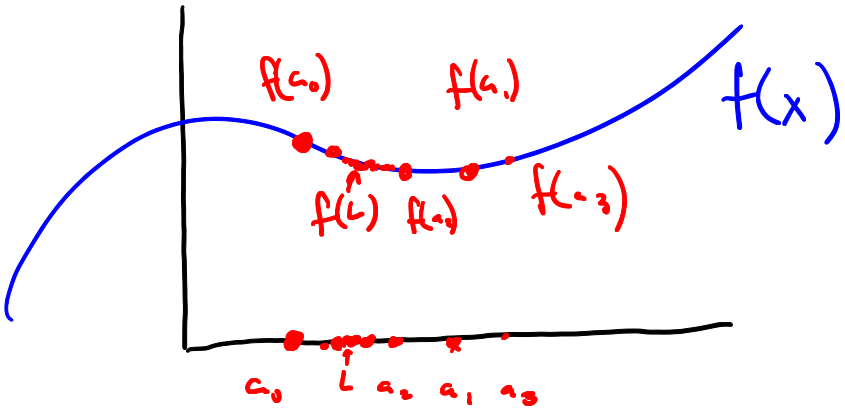
Ex 16 Why can't we use the Algebraic Limit Laws to directly compute $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$? ($\sqrt{n+1}\to\infty$, $\sqrt{n}\to\infty$)

Both $\sqrt{n+1}$ and $\sqrt{n}$ diverge to ∞

$$\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = 0$$

Thm: (Continuous Functions Defined on Convergent Sequences) Suppose that $\{a_n\}$ and L are in the domain of a function $f(x)$. Suppose that $f(x)$ is continuous at L and that $\lim_{n\to\infty} a_n = L$. Then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$



Ex 17 Compute $\lim_{n\to\infty}(\ln(2+3n)-\ln(n))$.

in domain of $\ln(x)$

$$\lim_{n\to\infty} \ln(2+3n) - \ln(n)$$

$$\lim_{n\to\infty} \ln\left(\frac{2+3n}{n}\right)$$

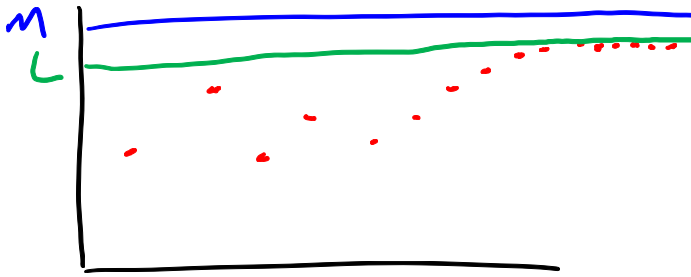
$$\frac{2+3n}{n} = \frac{2}{n} + 3 \xrightarrow[n\to\infty]{} 3$$

$$\ln\left(\lim_{n\to\infty} \frac{2+3n}{n}\right)$$

$$\ln(3)$$

Q. How are boundedness and monotonicity related to convergence?

Thm: (Monotone Convergence) If a sequence $\{a_n\}$ is bounded and monotone for all $n \geqslant n_0$, then a_n converges to some limit, L.



Ex 18 $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = ?$

$g_0 = 0 \qquad g_{n+1} = \sqrt{1+g_n}$

$g_0 = 0, \; g_1 = \sqrt{1+0}, \; g_2 = \sqrt{1+\sqrt{1}}, \; g_3 = \sqrt{1+\sqrt{1+\sqrt{1}}}$

Goal: $\lim_{n\to\infty} g_n$?

① Claim: g_n is monotone

$g_0 \leq g_1$

$1+g_0 \leq 1+g_1$

$\sqrt{1+g_0} \leq \sqrt{1+g_1}$ ← $\sqrt{\cdot}$ preserves inequalities

$g_1 \leq g_2$

$\sqrt{1+g_1} \leq \sqrt{1+g_2}$

$g_2 \leq g_3$

$\vdots$

$g_0 \leq g_1 \leq g_2 \leq g_3 \leq \cdots$ ✓

(2) Claim: $0 \le g_n \le 2$

$$0 \le g_0 \le 2$$

$$1 \le 1+g_0 \le 3$$

$$0 \le \sqrt{1} \le \sqrt{1+g_0} \le \sqrt{3} \le 2$$

$$0 \le g_1 \le 2$$

$$0 \le \sqrt{1} \le \sqrt{1+g_1} \le \sqrt{3} \le 2$$

$$0 \le g_2 \le 2$$

$$\vdots$$

$$0 \le g_n \le 2 \quad \checkmark$$

Conclude: By M.C.T. $\exists L$: $\lim_{n\to\infty} g_n = L$

$$g_{n+1} = \sqrt{1+g_n}$$

$$\updownarrow$$

$$g_{n+1}^2 = 1+g_n$$

$$\downarrow$$

$$L^2 = \left(\lim_{n\to\infty} g_{n+1}\right)^2 = \lim_{n\to\infty} g_{n+1}^2 = \lim_{n\to\infty} 1+g_n = 1+\lim_{n\to\infty} g_n = 1+L$$

$$L^2 = 1+L$$

$$L^2 - L - 1 = 0$$

$$L = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\to \boxed{\frac{1+\sqrt{5}}{2} = L}$$

$$\varphi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$

$$= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$$

← continued fraction